

The Poisson Bracket of Length functions in the Hitchin Component

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Abstract

Wolpert's cosine formula on Teichmüller space gives the Weil-Petersson Poisson bracket $\{l_\alpha, l_\beta\}$ for geodesic length functions l_α, l_β of closed curves α, β as the sum of the cosines of the angle of intersection of the associated geodesics. This was recently generalized to Hitchin representations by Labourie. In this paper, we give a short proof of this generalization using Goldman's formula for the Poisson bracket on representation varieties of surface groups into reductive Lie groups.

1 Introduction

Let S be a closed oriented surface of genus $g \geq 2$. In [2], Hitchin considered the space

$$\mathcal{R}_n(S) = \text{Hom}^{\text{red}}(\pi_1(S), \text{PSL}(n, \mathbb{R})) / \text{PSL}(n, \mathbb{R})$$

of conjugacy classes of reducible representations of $\pi_1(S)$ into $\text{PSL}(n, \mathbb{R})$. The space $\mathcal{R}_n(S)$ has the structure of algebraic variety.

In [4], Goldman showed that for $n = 2$, $\mathcal{R}_2(S)$ has $4g - 3$ components, two of which are the Teichmüller components $T(S), T(\bar{S})$ corresponding to the conformal structures on S and its complex conjugate \bar{S} respectively. The space $\mathcal{R}_n(S)$ has a natural symplectic structure ω , called the *Goldman symplectic form*, discovered by Goldman (see [5]). This generalized the symplectic form discovered by Atiyah-Bott for the case of representations into the group $U(n)$ (see [1]). For $n = 2$ the form ω restricts on $\mathcal{R}_n(S)$ to (an integer multiple of) the well-known Weil-Petersson symplectic form ω_{wp} on $T(S)$.

The symplectic form ω on $\mathcal{R}_n(S)$ defines a dual Poisson structure on $\mathcal{R}_n(S)$ given by $\{f, g\} = \omega(Hf, Hg)$ where Hf, Hg are the Hamiltonian vector fields with respect to ω of the smooth functions $f, g : \mathcal{R}_n(S) \rightarrow \mathbb{R}$.

Given α a homotopy class of a non-trivial closed curve on S , we have the associated length function $l_\alpha : T(S) \rightarrow \mathbb{R}$ which assigns the length of the geodesic representative of α in the associated hyperbolic structure. In [11], Wolpert showed that for the Weil-Petersson symplectic form, then $Hl_\alpha = -t_\alpha$ where t_α is the twist vector field obtained by dehn twist about α a simple non-trivial closed curve. Wolpert further proved the the following *cosine formula* for the Poisson bracket of length functions.

Theorem 1 (Wolpert, [10]) *Let $\{.,.\}_{wp}$ be the Poisson bracket on Teichmüller space $T(S)$ given by the Weil-Petersson symplectic form. Let α, β be homotopy classes of closed oriented curves in S with*

unique closed geodesic representatives $\overline{\alpha}, \overline{\beta}$ in $X \in T(S)$. Then

$$\{l_\alpha, l_\beta\}_{wp}(X) = \sum_{p \in \overline{\alpha} \cap \overline{\beta}} \cos \theta_p$$

where θ_p is the angle of intersection of $\overline{\alpha}, \overline{\beta}$ at p measured from $\overline{\alpha}$ to $\overline{\beta}$ counterclockwise.

As part of his proof of the Nielsen realization conjecture (see [6]), Kerkhoff also derived the above formula for the case when the curves are measured laminations.

In the recent preprint, *Goldman algebra, opers and the swapping algebra*, Labourie generalizes the above formula for Hitchin representations (see [7, Theorem 6.1.2]). In this note, we give another proof of this generalization using Goldman's formula for the Poisson bracket of invariant functions (see [3]).

A representation $\rho : \pi_1(S) \rightarrow \mathrm{PSL}(n, \mathbb{R})$ is *Hitchin* if there exists a Teichmüller representation $\rho_0 : \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{R})$ such that $\rho = \tau_n \circ \rho_0$ where $\tau_n : \mathrm{PSL}(2, \mathbb{R}) \rightarrow \mathrm{PSL}(n, \mathbb{R})$ is the irreducible representation. As Teichmüller space is connected, Hitchin representations correspond to (at most two) connected components of $R_n(S)$ given by the images of $T(S), T(\overline{S})$ under τ_n . Thus for $n = 2$ the Hitchin components are exactly the Teichmüller components $T(S), T(\overline{S})$. Hitchin proved the following;

Theorem 2 (Hitchin, [2]) *Each Hitchin component is homeomorphic to $\mathbb{R}^{|\chi(S)|(n^2-1)}$. If n is even there are exactly two Hitchin components and if n is odd, there is exactly one.*

Using techniques from the dynamics of Anosov flows, Labourie showed the following;

Theorem 3 (Labourie, [7]) *If ρ is a Hitchin representation then ρ is discrete faithful and for every $g \neq e$, $\rho(g)$ is diagonalizable over \mathbb{R} with eigenvalues distinct $\lambda_1(g), \dots, \lambda_n(g)$ satisfying*

$$|\lambda_1(g)| > |\lambda_2(g)| > \dots > |\lambda_n(g)|.$$

Thus given α a homotopy class or closed oriented curve in S , we therefore have functions $l_\alpha^i : H_n(S) \rightarrow \mathbb{R}$ given by

$$l_\alpha^i([\rho]) = \log |\lambda_i(\rho(\alpha))|.$$

In [9], Labourie introduced the following *cross-ratio* on quadruples of lines and planes. We let \mathbb{RP}^{n-1} be the space of lines in \mathbb{R}^n (considered as non-zero vectors in \mathbb{R}^n up to multiplication by \mathbb{R}^*), and \mathbb{RP}^{n-1*} the space of planes (considered as the space of non-zero linear functionals on \mathbb{R}^n up to multiplication by \mathbb{R}^*).

The cross-ratio is given by the map $b : \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1*} \times \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1*}$

$$b(x, y, z, w) = \frac{\langle y' | z' \rangle \langle w' | x' \rangle}{\langle y' | x' \rangle \langle w' | z' \rangle}$$

where $x' \in x, y' \in y, z' \in z, w' \in w$ are any choice of non-zero elements. By linearity b is well defined as the above formula is independent of the choices made. The cross-ratio b is obviously only defined when the quadruple (x, y, z, w) is in general position.

Given A a matrix with eigenvalues having distinct absolute values, we define $\xi^i(A) \in \mathbb{RP}^{n-1}$ to be the i -th eigenspace, and $\theta^i(A) \in \mathbb{RP}^{n-1*}$ to be the plane spanned by $\{\xi^j\}_{j \neq i}$. We let $\xi(A) = (\xi^1(A), \xi^2(A), \dots, \xi^n(A))$ and $\theta(A) = (\theta^1(A), \theta^2(A), \dots, \theta^n(A))$. We define

$$b^{ij}(A, B) = b(\xi^i(A), \theta^i(A), \xi^j(B), \theta^j(B)).$$

If $\rho : \pi_1(S) \rightarrow \mathrm{PSL}(n, \mathbb{R})$ is a Hitchin representation, and $\alpha, \beta \in \pi_1(S)$ then we define

$$b_\rho^{ij}(\alpha, \beta) = b^{ij}(\rho(\alpha), \rho(\beta)).$$

In [8], Labourie gives the following generalization of Wolpert's cosine formula.

Theorem 4 (Labourie, [8]) *Let α, β be homotopy classes of closed oriented curves in S represented by immersed curves $\overline{\alpha}, \overline{\beta}$ in S which are in general position, then*

$$\{l_\alpha^i, l_\beta^j\}([\rho]) = \sum_{p \in \overline{\alpha} \cap \overline{\beta}} \epsilon(p, \overline{\alpha}, \overline{\beta}) \left(b_{\rho_p}^{ij}(\overline{\alpha}_p, \overline{\beta}_p) - \frac{1}{n} \right).$$

We will give an elementary proof of this theorem.

We note that for $n = 2$ there is a single cross-ratio b and for $A, B \in \mathrm{PSL}(2, \mathbb{R})$, $b(A, B) = \cos^2(\phi_p/2)$ where ϕ_p is the angle of intersection between the positive rays of the associated geodesics in α, β in \mathbb{H}^2 at the point of intersection $p = \alpha \cap \beta$. Thus

$$b(A, B) - \frac{1}{2} = \frac{1}{2}(2 \cos^2(\phi_p/2) - 1) = \frac{1}{2} \cos(\phi_p).$$

The angle $\theta_p < \pi$ is the counterclockwise angle between α, β at their intersection point. Thus if $0 < \phi_p < \pi$, p is positively oriented then $\phi_p = \theta_p$ and if $\pi < \phi_p < 2\pi$ then p is negatively oriented and $\theta_p = \phi_p - \pi$. Thus the above formula for $n = 2$ is

$$\{l_\alpha^1, l_\beta^1\}([\rho]) = \sum_{p \in \overline{\alpha} \cap \overline{\beta}} \epsilon(p, \overline{\alpha}, \overline{\beta}) \left(\frac{1}{2} \cos(\phi_p) \right) = \frac{1}{2} \sum_{p \in \overline{\alpha} \cap \overline{\beta}} \cos(\theta_p).$$

For $g \in \mathrm{SL}(n, \mathbb{R})(2, \mathbb{R})$ we have $\lambda_1(g) = e^{l(g)/2}$ where $l(g)$ is the hyperbolic translation of g . Therefore it follows that if l_γ is the length function for closed curve γ then $l_\gamma = 2l_\gamma^1$. Also the classical Weil-Petersson symplectic form ω_{wp} satisfies $\omega = 2\omega_{wp}$ (see [3]). Therefore we recover Wolpert's cosine formula for the Weil-Petersson Poisson structure

$$\{l_\alpha, l_\beta\}_{wp} = \sum_{p \in \alpha' \cap \beta'} \cos \theta_p.$$

2 Background

We now describe the background on Goldman's formula for the Poisson bracket of invariant functions. Let G be a reductive matrix group and consider the non-degenerate symmetric form $\mathcal{B} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ given by $\mathcal{B}(X, Y) = \text{Tr}(XY)$. An invariant function for G is a smooth function $f : G \rightarrow \mathbb{R}$ which is conjugacy invariant. In particular $f = \text{Tr}$ is an invariant function. Given f there is a natural function $F : G \rightarrow \mathfrak{g}$ given by

$$\mathcal{B}(F(A), X) = \frac{d}{dt} f(\exp(tX)A) \quad \text{for all } X \in \mathfrak{g}$$

Thus $F(A)$ is dual to $R_A^*(df(A)) \in \mathfrak{g}^*$ under the isomorphism $\hat{\mathcal{B}} : \mathfrak{g} \rightarrow \mathfrak{g}^*$ given by $\hat{\mathcal{B}}(X)(Y) = \mathcal{B}(X, Y)$.

Let S be a closed oriented surface of genus $g \geq 2$ and $\pi = \pi_1(S, p)$ for some $p \in S$. We consider the space $\text{Hom}(\pi, G)/G$ of representations $\rho : \pi \rightarrow G$ up to conjugacy and let $\mathcal{R}(S, G)$ be the space of smooth points of $\text{Hom}(\pi, G)/G$. If α is a non-trivial homotopy class of closed oriented curve in S then α defines a conjugacy class in π . If f is an invariant function for G then we can define $f_\alpha : \mathcal{R}(S, G) \rightarrow \mathbb{R}$ by

$$f_\alpha([\rho]) = f(\rho(\alpha'))$$

where $\alpha' \in \alpha$.

The tangent space at $[\rho] \in \mathcal{R}(S, G)$ can be identified with the group cohomology $H^1(\pi, \mathfrak{g}_{\text{Ad} \circ \rho})$. Using \mathcal{B} to pair coefficients, we use the cup-product and cap-product for group cohomology to define the map

$$H^1(\pi, \mathfrak{g}_{\text{Ad} \circ \rho}) \times H^1(\pi, \mathfrak{g}_{\text{Ad} \circ \rho}) \xrightarrow{\mathcal{B}(\cup)} H^2(\pi, \mathbb{R}) \xrightarrow{\cap[\pi]} H_0(\pi, \mathbb{R}) = \mathbb{R}$$

This map defines the Goldman symplectic form ω on $\mathcal{R}(S, G)$ (see [5]). Specifically we have

$$\omega_{[\rho]}(X, Y) = \mathcal{B}(X \cup Y) \cap [\pi].$$

Given a smooth function $f : \mathcal{R}(S, G) \rightarrow \mathbb{R}$ the *Hamiltonian vector field* of f is the vector field Hf defined by $\omega(Hf, Y) = df(Y)$. For f, g two smooth functions the associated *Poisson bracket* on smooth functions is the pairing $\{., .\} : C^\infty(\mathcal{R}(S, G), \mathbb{R}) \times C^\infty(\mathcal{R}(S, G), \mathbb{R}) \rightarrow C^\infty(\mathcal{R}(S, G), \mathbb{R})$ given by

$$\{f, g\}([\rho]) = \omega_{[\rho]}(Hf, Hg).$$

Given α an oriented curve in S , if $p \in \alpha$, we let α_p be the oriented curve given by traversing α starting at p . If α, β are two oriented closed curves, then α, β are in general position if their intersections are transverse. If α, β are in general position, then for $p \in \alpha \cap \beta$ we define $\epsilon(p, \alpha, \beta) = \pm 1$ given by if the orientation of the point of intersection agrees or not with the orientation of the surface.

Also for $[\rho] \in \mathcal{R}(S, G)$ we let $\rho_p : \pi_1(S, p) \rightarrow G$ be a representation defined by change of base point of ρ . This is well-defined up to conjugacy.

Goldman gave the following description of the Poisson bracket for invariant functions.

Theorem 5 (Goldman, [3]) *Let $f, f' : G \rightarrow \mathbb{R}$ be invariant functions for G with associated functions $F, F' : G \rightarrow \mathfrak{g}$. Let α, β be homotopy classes of closed oriented curves represented by immersed curves*

$\overline{\alpha}, \overline{\beta}$ in S which are in general position. Then

$$\{f_\alpha, f'_\beta\}[\rho] = \sum_{p \in \overline{\alpha} \cap \overline{\beta}} \epsilon(p, \overline{\alpha}, \overline{\beta}) \mathcal{B}(F(\rho_p(\overline{\alpha}_p)), F'(\rho_p(\overline{\beta}_p)))$$

3 Length Functions

As Hitchin representations can be lifted to representations into $\mathrm{SL}(n, \mathbb{R})(n, \mathbb{R})$ (see [7]), we can restrict to representations into $\mathrm{SL}(n, \mathbb{R})(n, \mathbb{R})$. We define the hyperbolic elements $Hyp \subseteq \mathrm{SL}(n, \mathbb{R})$ to be the open subset of diagonalizable matrices over \mathbb{R} with eigenvalues having distinct absolute values. For $A \in Hyp$, A has eigenvalues $\lambda_1(A), \dots, \lambda_n(A)$ with $|\lambda_1(A)| > |\lambda_2(A)| > \dots > |\lambda_n(A)|$. We define the functions $l^i : Hyp \rightarrow \mathbb{R}$ by letting $L^i(A) = \log |\lambda_i(A)|$. We define the function $L^i : Hyp \rightarrow \mathfrak{sl}(n, \mathbb{R})$ by

$$\mathcal{B}(L^i(A), X) = \frac{d}{dt} l^i(\exp(tX)A).$$

3.1 Eigenvalue Perturbation

We now consider perturbation of eigenvalues in the space of hyperbolic matrices. Given $A \in Hyp$ let $p_i(A) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be projection onto the i -th eigenspace, parallel to the other eigenvectors.

Lemma 1 *The length function $l^i : Hyp \rightarrow \mathbb{R}$ satisfies*

$$dl_A^i(X) = \frac{1}{\lambda^i(A)} \mathrm{Tr}(p_i(A).X).$$

Proof: We let $A = A_0$ and denote the eigenvalues and eigenvectors of A by λ^i, x^i . We further let $\dot{A} = \dot{A}_0$. We have A_t has eigenvalues λ_t^i and unit length eigenvector x_t^i . We have

$$A_t.x_t^i = \lambda_t^i.x_t^i.$$

Differentiating we get

$$\dot{A}x^i + A\dot{x}^i = \dot{\lambda}^i.x^i + \lambda^i.\dot{x}^i$$

We let $p_i(A)$ be linear projection onto the i -th eigenspace of A parallel to the other eigenspaces of A . We apply to the above equation.

$$p_i(A).\dot{A}x^i + p_i(A)A\dot{x}^i = \dot{\lambda}^i.p_i(A)x^i + \lambda^i.\dot{x}^i$$

As $p_i(A)A = \lambda^i.p_i(A)$ we have $p_i(A)A\dot{x}^i = \lambda^i.p_i(A)\dot{x}^i$ so after cancellation we get

$$\dot{\lambda}^i.x^i = p_i(A).\dot{A}.x^i.$$

Therefore we have

$$\dot{\lambda}^i = \mathrm{tr}(p_i(A).\dot{A}).$$

As $l^i(X) = \log |\lambda^i(X)|$ on Hyp we have

$$dl^i = \frac{d\lambda^i}{\lambda^i}.$$

Therefore

$$dl_A^i(X) = \frac{1}{\lambda^i(A)} Tr(p_i(A).X)$$

□

We now use the above lemma to calculate L^i .

Lemma 2

$$L^i(A) = p_i(A) - \frac{1}{n}I.$$

Proof: By the above

$$\mathcal{B}(L^i(A), X) = \frac{d}{dt} l^i(\exp(tX)A) = dl_A^i(XA) = \frac{1}{\lambda^i(A)} Tr(p_i(A).XA).$$

By definition $A.p_i(A) = \lambda^i p_i(A)$. Therefore

$$\mathcal{B}(L^i(A), X) = \frac{1}{\lambda^i(A)} Tr(A.p_i(A).X) = \frac{1}{\lambda^i(A)} Tr(\lambda^i p_i(A).X) = Tr(p_i(A).X).$$

We let $\overline{\mathcal{B}} : \mathfrak{gl}(n, \mathbb{R}) \times \mathfrak{gl}(n, \mathbb{R}) \rightarrow \mathbb{R}$ given by $\mathcal{B}(X, Y) = Tr(XY)$. Then $\overline{\mathcal{B}}$ is non-degenerate and restricts to \mathcal{B} on \mathfrak{g} . We let $P : \mathfrak{gl}(n, \mathbb{R}) \rightarrow \mathfrak{sl}(n, \mathbb{R})$ be orthogonal projection with respect to $\overline{\mathcal{B}}$. Then given $A \in \mathfrak{gl}(n, \mathbb{R})$, then for all $X \in \mathfrak{g}$

$$\overline{\mathcal{B}}(A, X) = \overline{\mathcal{B}}(P(A), X) = \mathcal{B}(P(A), X).$$

Therefore we have

$$\mathcal{B}(L^i(A), X) = Tr(p_i(A).X) = \overline{\mathcal{B}}(p_i(A), X) = \mathcal{B}(P(p_i(A)), X).$$

As \mathcal{B} is non-degenerate on $\mathfrak{sl}(n, \mathbb{R})$, we have

$$L^i(A) = P(p_i(A))$$

The projection map $P : \mathfrak{gl}(n, \mathbb{R}) \rightarrow \mathfrak{sl}(n, \mathbb{R})$ is given by

$$P(A) = A - \frac{1}{n} Tr(A).I$$

Therefore as $p_i(A)$ is projection onto a 1-dimensional eigenspace, $Tr(p_i(A)) = 1$ and we have

$$L^i(A) = p_i(A) - \frac{1}{n} Tr(p_i(A)).I = p_i(A) - \frac{1}{n}.I$$

□

3.2 Poission bracket

We now use Goldman's formula to give an alternative proof of Labourie's generalization of the cosine formula.

Theorem 4 (*Labourie, [8]*)

$$\{l_\alpha^i, l_\beta^j\}([\rho]) = \sum_{p \in \bar{\alpha} \cap \bar{\beta}} \epsilon(p, \bar{\alpha}, \bar{\beta}) \left(b_{\rho_p}^{ij}(\bar{\alpha}_p, \bar{\beta}_p) - \frac{1}{n} \right).$$

Proof: From the above we have

$$\mathcal{B}(L^i(A), L^j(B)) = \text{Tr} \left(\left(p_i(A) - \frac{1}{n} I \right) \cdot \left(p_j(B) - \frac{1}{n} I \right) \right)$$

As $\text{Tr}(p_i(A)) = \text{Tr}(p_j(B)) = 1$

$$\mathcal{B}(L^i(A), L^j(B)) = \text{Tr}(p_i(A)p_j(B)) - \frac{1}{n}$$

Now applying Goldman's formula from Theorem 5 we get

$$\begin{aligned} \{l_\alpha^i, l_\beta^j\}([\rho]) &= \sum_{p \in \bar{\alpha} \cap \bar{\beta}} \epsilon(p, \bar{\alpha}, \bar{\beta}) \mathcal{B}(L^i(\rho(\bar{\alpha}_p)), L^j(\rho(\bar{\beta}_p))) \\ &= \sum_{p \in \bar{\alpha} \cap \bar{\beta}} \epsilon(p, \bar{\alpha}, \bar{\beta}) \left(\text{Tr}(p_i(\rho(\bar{\alpha}_p))p_j(\rho(\bar{\beta}_p))) - \frac{1}{n} \right). \end{aligned}$$

For any $X \in Hyp$ and let $\xi(X), \theta(X)$ be the n -tuples of eigenspaces and dual planes. We let $A, B \in Hyp$ and we choose non-zero elements $a_+^i \in \xi^i(A), a_-^i \in \theta^i(A), b_+^j \in \xi^j(B), b_-^j \in \theta^j(B)$. Then

$$p_i(A)(v) = \frac{\langle a_-^i | v \rangle}{\langle a_-^i | a_+^i \rangle} a_+^i \quad p_j(B)(v) = \frac{\langle b_-^j | v \rangle}{\langle b_-^j | b_+^j \rangle} b_+^j.$$

Similarly for $B \in Hyp$ with b_i^+, b_i^- . Then if $A, B \in Hyp$ we have

$$p_i(A)p_j(B)v = \frac{\langle a_-^i | b_+^j \rangle}{\langle a_-^i | a_+^i \rangle} \frac{\langle b_-^j | v \rangle}{\langle b_-^j | b_+^j \rangle}$$

Thus

$$\text{Tr}(p_i(A)p_j(B)) = \frac{\langle a_-^i | b_+^j \rangle \langle b_-^j | a_+^i \rangle}{\langle a_-^i | a_+^i \rangle \langle b_-^j | b_+^j \rangle} = b(\xi^i(A), \theta^i(A), \xi^j(B), \theta^j(B)) = b^{ij}(A, B).$$

Therefore the Poisson bracket is

$$\{l_\alpha^i, l_\beta^j\}([\rho]) = \sum_{p \in \bar{\alpha} \cap \bar{\beta}} \epsilon(p, \bar{\alpha}, \bar{\beta}) \left(b_{\rho_p}^{ij}(\bar{\alpha}_p, \bar{\beta}_p) - \frac{1}{n} \right).$$

□

References

- [1] M. Atiyah, R. Bott “The Yang-Mills equations over Riemann surfaces,” *Phil. Trans. R. Soc. London Ser., A* **308** (1983), 523-615.
- [2] N. Hitchin, “Lie groups and Teichmüller space,” *Topology* **31**(1992), 449–473.
- [3] W. Goldman, “Invariant functions on Lie groups and Hamiltonian flows of surface group representations,” *Invent. Math.* **85**(1986), 263-302.
- [4] W. Goldman, “Topological components of spaces of representations,” *Invent. Math.*, **93**, 557-607 (1988)
- [5] W. Goldman, “The symplectic nature of fundamental groups of surfaces,” *Adv. Math.* **54** (1984), no. 2, 200-225.
- [6] S. Kerckhoff “The Nielsen realization problem,” *Annals of Math.* **117** (**2**) (1983), no. 2, pp. 235–265.
- [7] F. Labourie, “Anosov flows, surface groups and curves in projective space,” *Invent. Math.* **165**(2006), 51–114.
- [8] F. Labourie, “Goldman algebra, opers and the swapping algebra,” preprint (2012), arXiv:1212.5015
- [9] F. Labourie. “Cross Ratios, Surface Groups, $SL_n(\mathbb{R})$ and Diffeomorphisms of the Circle,” *Publ. Math. de l’I.H.E.S.* **106**(2007), 139–213.
- [10] S. Wolpert, “ An elementary formula for the Fenchel-Nielsen twist,” *Comm. Math. Helv.*, **56**, 132-135 (1981)
- [11] S Wolpert, “The Fenchel-Nielsen deformation,” *Ann. of Math.*, **2**, 115 (1982), no. 3, 501-528.